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ASYMPTOTICS FOR THE HEAT KERNEL IN MULTICONE DOMAINS

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Abstract. A multi cone domain $\Omega \subseteq \mathbb{R}^n$ is an open, connected set that resembles a finite collection of cones far away from the origin. We study the rate of decay in time of the heat kernel $p(t,x,y)$ of a Brownian motion killed upon exiting Ω, using both probabilistic and analytical techniques. We find that the decay is polynomial and we characterize $\lim_{t\to\infty} t^{1+\alpha}p(t,x,y)$ in terms of the Martin boundary of Ω at infinity, where $\alpha > 0$ depends on the geometry of Ω. We next derive an analogous result for $t^{\kappa/2}\mathbb{P}_x(T > t)$, with $\kappa = 1 + \alpha - n/2$, where T is the exit time form Ω. Lastly, we deduce the renormalized Yaglom limit for the process conditioned on survival.

1. Introduction. Let O be a domain (open and connected set) in $\mathbb{R}^n$, regular for the Dirichlet problem. Consider an $n-$dimensional Brownian motion B_t starting from the interior of O, with exit time T^O. The heat kernel $p^O(t,x,y)$ is the Radon-Nikodym derivative of the Borel measure $A \mapsto \mathbb{P}_x(B_t \in A, T^O > t)$ with respect to the $n-$dimensional Lebesgue measure, and it is characterised to be the fundamental solution of the heat equation with Dirichlet boundary condition, that is: as a function of (t,y) it solves the heat equation $\partial_t u = \frac{1}{2}\Delta u$, it vanishes continuously on ∂O, and it satisfies the initial condition $u(0,y) = \delta_x(y)$.

It is well known that $p^O(t,x,y)$ tends to zero as time grows to infinity. A classical problem is to find the exact asymptotic (in time) for the decay of the heat kernel and the survival probability. This is well understood for bounded domains (see [10] and [11]). For results in some planar domains we refer the

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reader to [2]. The large time asymptotic problem is treated in [9] for a large class of (non symmetric) diffusions under some integrability conditions on the ground state. Exact asymptotic are computed for Benedicks domains in [4], and for exterior domains in [5]. Our work focuses on finding the exact asymptotic in time for $p^{\Omega}(t,x,y)$ and $\mathbb{P}_x(T^{\Omega} > t)$ for a multicone domain Ω, which we define next.

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Points in $\mathbb{R}^n$ will be regarded as $x = r\theta$, where $r = |x|$ and $\theta \in \mathbb{S}^{n-1}$. Given a Lipschitz, proper subdomain $\mathfrak{D}$ of $\mathbb{S}^{n-1}$, and a vector $a \in \mathbb{R}^n$, a truncated cone with opening $\mathfrak{D}$ and vertex a is the set

$$C(a, \mathfrak{D}, R) = \{a + x : x = r\theta \in \mathbb{R}^n : r > R,\ \theta \in \mathfrak{D}\},$$

where $R \geq 0$. When $R > 0$, the set $\mathfrak{S} = a + R\mathfrak{D}$ will be called the *base* of the truncated cone. When $R = 0$, we will refer to the set in the previous display as cone with vertex a.

In the same context as above, given a base $\mathfrak{S} = a + R\mathfrak{D}$, let $0 < \lambda^1 < \lambda^2 \leq \lambda^3 \leq \cdots$ be the eigenvalues of the Laplace-Beltrami operator on $\mathfrak{D}$, with corresponding orthonormal basis $\{m^1, m^2, m^3, \ldots\}$ of $L^2(\mathfrak{D}, \sigma)$, where σ is the surface measure on $\mathbb{S}^{n-1}$. Let $\alpha^i = \left(\lambda^i + (\frac{n}{2} - 1)^2\right)^{1/2}$. We define the character of the base $\mathfrak{S}$ as the number $\alpha = \alpha(\mathfrak{D}) = \alpha^1$. The character of the truncated cone $C(a, \mathfrak{D}, R)$ is also defined as α.

A multicone domain $\Omega \subseteq \mathbb{R}^n$ is a connected, open set such that there exists a bounded domain $\Omega_0 \subseteq \Omega$ and finitely many truncated cones $\Omega_j = C(a_j, \mathfrak{D}_j, R_j)$, with $j = 1, \ldots N$, such that $\Omega_j \cap \Omega_i = \emptyset$ for $1 \leq j < i \leq N$, and

$$\Omega \setminus \overline{\Omega}_0 = \bigcup_{j=1}^{N} \Omega_j.$$

Here $\overline{\Omega}_0$ is the closure of the set Ω_0. The set Ω_0 will be called the *core*, and for $j \geq 1$, the sets Ω_j are called *branches* of the multi-cone set. Notice that by construction, the branches are disjoint from the core. Also, we will denote the base of the truncated cone Ω_j by $\mathfrak{S}_j$. Without loss of generality, we can assume that $R_j = 1$, which makes the exposition that follows much easier. The character of the truncated cone Ω_j will be denoted by α_j. We define the character of the multicone Ω as the number $\alpha = \min\{\alpha_j : j = 1, \ldots, N\}$. An index l such that $\alpha_l = \alpha$ will be called maximal. We denote by $\mathcal{M}$ the set of maximal indices.

To state the main results of this article, we need to introduce the Martin boundary at infinity for Ω.

It is well known that there is a unique minimal harmonic function w on a cone with vertex $\mathcal{C}_0 = C(a, \mathfrak{D}, 0)$ that vanishes continuously on $\partial\mathcal{C}_0$. Actually, there is only one positive harmonic function in $\mathcal{C}_0$ that vanishes continuously on its boundary (Theorem 1.1 in [1]). For $x = a + |x-a|\,\theta \in \mathcal{C}_0$ this function is given by:

$$v(x) = |x-a|^{\alpha - \left(\frac{n}{2}-1\right)} m^1(\theta), \tag{1.1}$$

where α is the character of $\mathfrak{D}$ and m^1 is the first eigenfunction of the Laplace-Beltrami operator on $\mathfrak{D}$. Notice how we have chosen to normalize w in terms of the normalization of m^1 in $L^2(\mathfrak{D}, \sigma)$. In order to simplify our exposition, we set $\kappa = 1 + \alpha - n/2$, so that $v(x) = |x-\alpha|^\kappa\, m^1(\theta)$.

Similarly, if $\mathcal{C} = C(a, \mathfrak{D}, R)$ is a truncated cone, there is a unique (minimal) positive harmonic function w in $\mathcal{C}$ that vanishes continuously on $\partial\mathcal{C}$, which is defined as follows: let $T^{\mathcal{C}}$ be the exit time of a Brownian motion B_t from the cone $\mathcal{C}$. Then

$$w(x) = v(x) - \mathbb{E}_x(v(B_{T^{\mathcal{C}}})), \qquad x \in \mathcal{C}. \tag{1.2}$$

Let w_j be the unique minimal harmonic function in Ω_j. By a standard balayage argument [7], one can extend w_j to a minimal harmonic function in Ω. Such extension is given by

$$u_j(x) = w_j(x)\mathbb{1}_{\Omega_j}(x) + \frac{1}{2}\int_{\mathfrak{S}_j} G(x,y)\partial_n w_j(y)\sigma_j(dy), \qquad x \in \Omega, \tag{1.3}$$

where ∂_n denotes the (inward) normal derivative on $\mathfrak{S}_j$, and σ_j is the translation of σ by a_j, and G is the Green function of the domain Ω:

$$G(x,y) = \int_0^\infty p(t,x,y)dt.$$

Reciprocally, we have that

$$w_j(x) = u_j(x) - \mathbb{E}_x u(B_{T^j}), \qquad x \in \Omega_j, \tag{1.4}$$

where B is an $n-$dimensional Brownian motion, stopped at its exit time T^j from Ω_j.

It is direct to verify from the last two equations that the function u_j is bounded in $\Omega \setminus \Omega_j$, and satisfies that for $x = a_j + r\theta$,

$$\lim_{r\to\infty} \frac{u_j(a_j + r\theta)}{w_j(a_j + r\theta)} = 1, \tag{1.5}$$

for fixed $\theta \in \mathfrak{D}_j$.

We are ready to state the main results of this paper.

THEOREM 1.1. *Let Ω be a multicone domain with branches $\Omega_1, \ldots, \Omega_N$. Let $\alpha > 0$ be the character of Ω, and let $\mathcal{M}$ be the set of maximal indices. Then,*

$$\lim_{t\to\infty} t^{1+\alpha} p(t,x,y) = \frac{1}{2^\alpha \Gamma(1+\alpha)} \sum_{l\in\mathcal{M}} u_l(x) u_l(y), \tag{1.6}$$

The limit is in the topology of uniform convergence on compact sets.

THEOREM 1.2. *Let Ω be a multicone domain with branches $\Omega_1, \ldots, \Omega_N$. Let $\alpha > 0$ be the character of Ω, and let $\mathcal{M}$ be the set of maximal indices. Set $\kappa = 1 + \alpha - n/2$. Then*

$$\lim_{t\to\infty} t^{\kappa/2} \mathbb{P}_x(T > t) = \frac{\Gamma\left(\frac{\kappa+n}{2}\right)}{2^{\kappa/2}\Gamma\left(\kappa + \frac{n}{2}\right)} \sum_{l\in\mathcal{M}} \left(\int_{\mathfrak{D}_l} m_l^1(\theta)\sigma(d\theta) \right) u_l(x). \tag{1.7}$$

The limit is in the topology of uniform convergence on compact sets.

THEOREM 1.3. *Let Ω be a multicone domain with character $\alpha > 0$, and set $\beta = 1 + \alpha + n/2$. Fix $x \in \Omega$, and $1 \le j \le N$. For each $y = |y|\,\theta$, with $\theta \in \mathfrak{D}_j$, we have that $a_j + \sqrt{t}y \in \Omega_j$, for large enough values of t, and*

$$\lim_{t\to\infty} t^{\beta/2} p(t,x,a_j + \sqrt{t}y) = \mathbb{1}_{\mathcal{M}}(j) \frac{u_j(x) v_j(y)}{2^\alpha \Gamma(1+\alpha)} e^{-|y|^2/2}. \tag{1.8}$$

The limit is in the sense of uniform convergence on compact sets on the variables x and y.

The paper is organized as follows. Section 2 lists some key results that we take from the literature on heat kernels for killed diffusions, in particular, subsection 2.1 includes our main theorems for the case of a cone with vertex. Section 3 deals with the asymptotics for truncated cones, and Section 4 includes some lemmas leading up to the proofs of the main theorems, which are contained at the end of Section 4 for the decay of the heat kernel, and in Section 5 for the decay of the survival probability. Finally, Section 6 includes the proof of Theorem 1.3 and discusses a renormalized Yaglom limit for the killed Brownian motion.

2. Preliminary results. In what follows we make the following simplifications, in order to keep the exposition clear. We set $T = T^{\Omega}$, $T^j = T^{\Omega_j}$ and denote by p and p^j the respective heat kernels. In some of the formulas below, integrals over $\mathfrak{S}_j$ are understood to be with respect to the translated measure σ^j, but we will omit the index since the dependence on j is clear from the domain of integration. Also, we will abuse the notation by omitting the vector a_j form all the formulas involving functions in cones, since its inclusion affects all such functions by a simple translation of coordinates. In particular, we will write $p^j(t,x,y)$ for $x = |x|\,\theta, y = |y|\,\eta$ for $\theta, \eta \in \mathfrak{D}_j$ instead of $p^j(t, x + a_j, y + a_j)$ in order to simplify our exposition. In this spirit, we will often say that $x \to \infty$ radially in Ω_j to mean that $x = a_j + r\theta$, and $r \to \infty$.

We start by listing some general properties of heat kernels in unbounded domains.

Lemma 2.1 (Lemma 2.1 in [5]). *Let O be a regular domain for the Dirichlet problem. Let $u(t,x)$ be a positive solution of the heat equation in $\mathbb{R}_+ \times O$, and consider a function $a : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\sup_{t \geq t_0, |s| \leq 2} \frac{a(t+s)}{a(t)} < \infty, \tag{2.1}$$

for some $t_0 > 0$. Further, assume that the family of functions $\{a(t)u(t,\cdot) : t \geq t_0\}$ is bounded on compact sets. Then, the family $\{a(t)u(t,\cdot) : t \geq t_0 + 1\}$ is equicontinuous on compact sets of O.

The next lemma corresponds to Lemmas 2.1-2.4 in [4], which are proved for Benedicks domains in $\mathbb{R}^n$. Nonetheless, the proofs work in a much more general setting, as long as the domain O is a regular domain for the Dirichlet problem, with infinite interior radius.

Lemma 2.2 (Lemmas 2.1-2.4 in [4]). *In the same setting of Lemma 2.1, for $x, y \in O$ and $s \in \mathbb{R}$ we have*

$$\lim_{t \to \infty} \frac{p(t+s,x,y)}{p(t,x,y)} = 1. \tag{2.2}$$

The limit is uniform in compact sets of $\overline{O}$. Also, the map $t \mapsto p(t,x,x)$ is decreasing.

LEMMA 2.3. *In the same setting as in Lemma 2.2, further assume that for all $s \in \mathbb{R}$,*

$$\lim_{t\to\infty} \frac{a(t+s)}{a(t)} = 1. \tag{2.3}$$

If $a(t)p(t,x,y) \leq C_x^{1+|y|}$ for large enough t, then any limit point of $a(t)p(t,\cdot,\cdot)$ (in the topology of uniform convergence on compact sets) has the following properties:

(i) is a symmetric, non-negative function;
(ii) is harmonic in each component;
(iii) and vanishes continuously on ∂O.

PROOF. For the sake of simplicity we denote $h_t(x,y) = a(t)p(t,x,y)$. Let $t_k \to \infty$ be a sequence such that h_{t_k} converges uniformly on compact sets of O to a function h. It is clear that h is symmetric and non-negative. Notice that for any $s \in \mathbb{R}$, the sequence h_{t_k+s} also converges uniformly on compact sets of O. This is direct from Lemma 2.2 and the hypothesis.

By the Chapman-Kolmogorov equation, for any $s \in \mathbb{R}$ and large enough $k \in \mathbb{N}$,

$$h_{t_k+s}(x,y) = \frac{a(t_k+s)}{a(t_k)} \int_\Omega h_{t_k}(x,z)p(s,z,y)dz.$$

By assumption, $h_{t_k}(x,z) \leq C_x^{1+|z|}$, which is $p(s,z,y)dz$-integrable as it can be checked by comparing p with the free Brownian motion's kernel. Thus, we can apply the Dominated Convergence Theorem to obtain

$$h(x,y) = \int_\Omega h(x,z)p(s,z,y)dz = \mathbb{E}_y(h(x,X_s)).$$

It is standard to show that $h(x,X_s)$ is a martingale, from where its standard to deduce that $y \mapsto h(x,y)$ is harmonic by means of the optional sampling theorem.

Consider a sequence $y_n \in O$, with $y_n \to y \in \partial O$. By using once again the Gaussian upper bound on p, and applying the Dominated Convergence Theorem to $h(x,z)p(1,z,y_n)$, it is deduced that $h(x,\cdot)$ vanishes continuously on ∂O.

$\square$

LEMMA 2.4. *Let U and O be domains in $\mathbb{R}^n$ that are regular for the Dirichlet problem. For $\xi \in \partial U$, $x \in U$*

$$\mathbb{P}_x(B_{T^U} \in \sigma(d\xi), T^U \in ds) = \frac{1}{2}\partial_n p^U(s,x,\xi)\sigma(d\xi)ds. \tag{2.4}$$

Here, ∂_n represents the inward normal derivative at $\xi \in \partial U$.

Also, if $U \subseteq O$, then

$$p^O(t,z,y) = p^U(t,z,y) + \int_0^t \int_{\partial U} \frac{1}{2}\partial_n p^U(s,x,\xi)p^O(t-s,\xi,y)\sigma(d\xi)ds. \tag{2.5}$$

PROOF. These results are well known so we only are going to comment their proofs. The proof of (2.4) uses Green's theorem and the heat equation, and it is very straightforward carry out. Equation (2.5) follows as an elementary application of the strong Markov property at time T^U. □

The following lemma characterizes all positive, harmonic functions vanishing on $\partial\Omega$. In other words, we characterise the Martin boundary of Ω. We use the notation from the Introduction.

LEMMA 2.5. *Let $u_1, \ldots, u_N$ be the minimal harmonic functions given by (1.3). For every nonnegative harmonic function u in Ω, vanishing continuously on $\partial\Omega$, there are unique nonnegative coefficients $\gamma_1, \cdots, \gamma_N$ such that*

$$u(x) = \sum_{j=1}^{N} \gamma_j u_j(x), \qquad x \in \Omega. \tag{2.6}$$

PROOF. For $x \in \Omega_j$, consider the harmonic function $\tilde{w}_j(x) = u(x) - \mathbb{E}_x(u(B_{T^j}))$. It is standard to check that $\tilde{w}_j$ is harmonic in Ω_j, and that vanishes continuously on $\partial\Omega_j$. For $m > R$, let T^j_m be the exit time from the set $\Omega_j \cap B(a_j, m)$. By Itô's formula, the process $u(B_{t\wedge T^j_m})$ is a bounded martingale under $\mathbb{P}_x$, for $x \in \Omega_j$. Therefore,

$$\begin{aligned} u(x) = \mathbb{E}_x\left(u(B_{T^j_m})\right) &= \mathbb{E}_x\left(u(B_{T^j_m})\mathbb{1}_{\{T^j_m < T^j\}}\right) + \mathbb{E}_x\left(u(B_{T^j})\mathbb{1}_{\{T^j_m = T^j\}})\right) \\ &\geq \mathbb{E}_x\left(u(B_{T^j})\right) - \mathbb{E}_x\left(u(B_{T^j})\mathbb{1}_{\{T^j_m < T^j\}})\right). \end{aligned}$$

Since $T_m^j \nearrow T^j$, monotone convergence shows that $u(x) \geq \mathbb{E}_x\left(u(B_{T^j})\right)$, that is, $\tilde{w}_j$ is nonnegative. Thus, $\tilde{w}_j(x) = \gamma_j w_j(x)$ by uniqueness. For $z \in \Omega$, set

$$\tilde{u}(z) = \sum_{j=1}^{N} \gamma_j u_j(z) - u(z),$$

which is harmonic in Ω, and vanishes continuously on ∂O. We will next show that $\tilde{u}$ is bounded, for which it is enough to show that it is bounded in each branch of Ω.

Fix $i \in \{1, \ldots, N\}$, and consider $x \in \Omega_i$. We have

$$\tilde{u}(x) = -\mathbb{E}_x\left(u(B_{T^i})\right) + \gamma_i \mathbb{E}_x\left(u_i(B_{T^i})\right) + \sum_{j=1, j\neq i}^{N} \gamma_j u_j(x)$$

The first term on the right hand side is bounded by $\sup_{x\in\Gamma_i} |u(x)|$, and the second one by $\gamma_i \sup_{x\in\Gamma_i} |u_i(x)|$. The summation is bounded as each term $u_j(x)$ is bounded in Ω_i. We conclude that $\tilde{u}$ is harmonic and bounded in Ω, and vanishes continuously on $\partial\Omega$. It follows that $\tilde{u}\left(B_{t\wedge T}\right)$ is a martingale, and so

$$\tilde{u}(z) = \mathbb{E}_z\left(\tilde{u}\left(B_{t\wedge T}\right)\right) \to 0, \text{ as } t \to \infty.$$

Uniqueness follows from the boundedness of u_j in $\Omega\backslash\Omega_j$, and its unboundedness in Ω_j. $\square$

2.1. *Asymptotics in a cone with vertex.* In what follows we consider a cone V, with opening $\mathfrak{D}$ and vertex $a = 0$, that is, $V = C(0, \mathfrak{D}, 0)$. Let p^V be the heat kernel in V. Let $0 < \lambda^1 < \lambda^2 \leq \lambda^3 \leq \cdots$ be the eigenvalues of the Laplace-Beltrami operator on $\mathfrak{D}$, with corresponding orthonormal basis $\{m^1, m^2, m^3, \ldots\}$ of $L^2(\mathfrak{D}, \sigma)$. We also denote by $\alpha^i = \left(\lambda^i + (\frac{n}{2} - 1)^2\right)^{1/2}$.

The behaviour of the heat kernel with Dirichlet boundary conditions is well known for a cone with vertex. The following results are taken from [3].

Theorem 2.6. *For $x = r\theta, y = \rho\omega \in V$, with $\theta, \omega \in \mathfrak{D}$ and $r = |x|$, $\rho = |y|$, the heat kernel with Dirichlet boundary conditions in V is given by:*

$$p^V(t, x, y) = \frac{\exp\left(-\frac{r^2+\rho^2}{2t}\right)}{t\,(r\rho)^{\frac{n}{2}-1}} \sum_{i=1}^{\infty} J_{\alpha^i}\left(\frac{r\rho}{t}\right) m^i(\theta) m^i(\omega) \tag{2.7}$$

where J_ν is the modified Bessel function of first kind of order ν, that is, the solution of

$$z^2 J_\nu''(z) + zJ_\nu' - (z^2+\nu^2)J_\nu = 0,$$

satisfying the growing conditions:

$$\frac{z^\nu}{2^\nu\Gamma(1+\nu)} \le J_\nu(z) \le \frac{z^\nu}{2^\nu\Gamma(1+\nu)}e^z, \tag{2.8}$$

for $z>0$, and $\nu \ge 0$.

Recall that the unique minimal positive harmonic function in V is given by $v(x) = v(|x|\,\theta) = |x|^{\alpha^1-(\frac{n}{2}-1)}\,m^1(\theta)$.

Corollary 2.7. *For each $x,y\in V$, we have*

$$\lim_{t\to\infty} t^{1+\alpha^1} p^V(t,x,y) = \frac{v(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)} \tag{2.9}$$

$$\lim_{t\to\infty} \frac{p^V(t,x,y)}{p^V(t,w,z)} = \frac{v(x)v(y)}{v(w)v(z)} \tag{2.10}$$

Both limits are uniform in compact sets.

Proof. Clearly, (2.10) follows from (2.9), so we only prove the latter. From Theorem 2.6, we get the bound

$$\left| t^{1+\alpha^1} p^V(t,x,y) - \frac{v(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}\right| \le C\left| \frac{t^{\alpha^1} e^{-\frac{r^2+s^2}{2t}}}{(rs)^{\frac{n}{2}-1}} J_{\alpha^1}\left(\frac{rs}{t}\right) - \frac{(rs)^{\alpha^1-(\frac{n}{2}-1)}}{2^{\alpha^1}\Gamma(1+\alpha^1)}\right| + \\ + t^{-(\alpha^2-\alpha^1)} \sum_{k=2}^{\infty} \frac{(rs)^{\alpha_k-(\frac{n}{2}-1)}}{2^{\alpha_k}\Gamma(1+\alpha_k)},$$

where $C = \sup_{\theta\in\mathfrak{D}} m^1(\theta)^2$. The uniform convergence on compact sets for the first term is easily deduced from (2.8). The series on the right hand side converges uniformly in compact sets, so the whole term converges to zero, as $t\to\infty$, since $\alpha^2 > \alpha^1$. □

3. Asymptotics in a truncated cone. The main goal of this section is to extend Corollary 2.7 to a truncated cone $\mathcal{C} = C(a,\mathfrak{D},R)$. As before, we assume that $R=1$ and $a=0$.

We will often use the following version of the Harnack inequality up to the boundary.

THEOREM 3.1 (From [12], see also [8]). *Let O be a precompact, regular domain for the Dirichlet problem, and let $u \geq 0$ be a solution of the heat equation on $O \times [0,T)$ with Dirichlet boundary condition. Then, given $x \in O$, there is $C_1 > 0$ such that $u(t,z) \leq C_1 u(T,x)$, for all $(t,z) \in [0,T) \times \overline{O}$ where the constant C_1 depends only on x and $T-t$.*

COROLLARY 3.2. *Let V be a cone with vertex. For any $x \in V$ there is a constant $C_x > 0$, only dependent on x, such that for all $y \in V$ the following inequality holds for all $t > 1$:*

$$p^V(t,x,y) \leq C_x^{1+|y|} p^V(t,x,x). \tag{3.1}$$

PROOF. Assume $|x| = 1$, otherwise the corollary follows by scaling. The inequality holds for small $|y|$, by a direct application of the boundary Harnack inequality (Theorem 3.1), so we assume that $|y| > 2$.

Let r be positive, but small enough so that $B(x,r) \subseteq V$. It follows by scaling that $B(\nu x, r) \subseteq V$ for all $\nu \geq 1$. Thus, applying the standard parabolic Harnack inequality several times in the ball $B(0,r)$ to the function $u(s,z) = p^V(t+s, \nu x + z, y)$ for fixed, but arbitrary $\nu > 1, y \in V$, we get

$$p^V(t,\nu x,y) \leq C_2^{1+r\nu} p^V(t+1+r\nu, x, y) \leq C_2^{2+2r\nu} p^V(t+2+2r\nu, \nu x, y),$$

for a positive constant C_2 that only depends on x.

The heat kernel in V has the following scaling property:

$$p^V(t,x,y) = \lambda^{-n} p^V\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}, \frac{y}{\lambda}\right), \qquad \lambda > 0.$$

From all the inequalities above, it follows that

$$\begin{aligned} p^V(t,x,y) &\leq C_3^{1+|y|} p^V(t+1+r|y|, x|y|, y) \\ &= C_3^{1+|y|} |y|^{-n} p^V\left(\frac{t+1+r|y|}{|y|^2}, x, \frac{y}{|y|}\right) \\ &\leq C_1 C_3^{1+|y|} |y|^{-n} p^V\left(\frac{t+1+r|y|}{|y|^2}+1, x, x\right) \\ &\leq C_1 C_3^{1+|y|} |y|^{-n} p^V\left(\frac{t}{|y|^2}, x, x\right), \end{aligned}$$

where the second to last line comes from the boundary Harnack inequality, whereas the last one comes form the fact that $t \mapsto p^V(t,x,x)$ is decreasing (see Lemma 2.2). Applying scaling once again,

$$\begin{aligned} p^V(t,x,y) &\leq C_1 C_3^{1+|y|} p^V\left(t, x\,|y|\,, x\,|y|\right) \\ &\leq C_1 C_3^{3+3|y|} p^V\left(t,x,x\right), \end{aligned}$$

as desired. □

LEMMA 3.3. *Let $\mathfrak{C} = C(a,\mathfrak{D},1)$, $\mathfrak{S} = a+\mathfrak{D}$, and $V = C(a,\mathfrak{D},0)$. There is a universal constant $Q > 0$ such that for any $x \in \mathfrak{C}$, and $\xi \in \mathfrak{S}$,*

$$\limsup_{t\to\infty} \frac{\partial_n p^{\mathfrak{C}}(t,x,\xi)}{p^V(t,x,x)} \leq \frac{Q}{v(x)}, \tag{3.2}$$

where v is the unique minimal harmonic function in V, normalized as in (1.1).

PROOF. By a translation of coordinates, we can assume $a = 0$. Set $U = B(0,1)^c$. By monotonicity of domains, and since both $p^{\mathfrak{C}}(t,x,\cdot)$ and $p^U(t,x,\cdot)$ vanish on $\mathfrak{S}$, we have that $0 \leq \partial_n p^{\mathfrak{C}}(t,x,\xi) \leq \partial_n p^U(t,x,\xi)$. Recall that there are constants $A > 0, B > 0$ such that $\partial_n p^U(1,x,\xi) \leq A\exp(-B\,|x|^2)$, so $0 \leq \partial_n p^{\mathfrak{C}}(1,x,\xi) \leq A\exp(-B\,|x|^2)$ for $\xi \in \mathfrak{S}$. These bounds allows us to compute the normal derivative from the Chapman-Kolmogorov equation as follows

$$\begin{aligned} \partial_n p^{\mathfrak{C}}(t+1,x,\xi) &= \int_{\mathfrak{C}} p^{\mathfrak{C}}(t,x,z)\partial_n p^{\mathfrak{C}}(1,z,\xi)dz \\ &\leq \int_{\mathfrak{C}} p^V(t,x,z)\partial_n p^{\mathfrak{C}}(1,z,\xi)dz. \end{aligned} \tag{3.3}$$

Thus,

$$\frac{\partial_n p^{\mathfrak{C}}(t+1,x,\xi)}{p^V(t,x,x)} \leq \int_{\mathfrak{C}} \frac{p^V(t,x,z)}{p^V(t,x,x)} \partial_n p^{\mathfrak{C}}(1,z,\xi)dz.$$

We intend to apply the Dominated Convergence Theorem to the integral on the right hand side. Equation (2.10) shows pointwise convergence as $t\to\infty$, and Corollary 3.2 together with the remarks at the beginning of this proof show that the integrand is dominated. Therefore

$$\limsup_{t\to\infty} \frac{\partial_n p^{\mathfrak{C}}(t+1,x,\xi)}{p^V(t,x,x)} \leq \frac{1}{v(x)} \int_{\mathfrak{C}} v(z)\partial_n p^{\mathfrak{C}}(1,z,\xi)dz.$$

The integral can be estimated using the explicit formula for $v(z)$, and the bound for $\partial_n p^{\mathfrak{C}}(1,z,\xi)$ discussed at the beginning of this proof. Finally, we use Lemma 2.2 to conclude.

□

THEOREM 3.4. *Let V a cone with opening $\mathfrak{D}$ and vertex a, and let its truncated version be $\mathcal{C} = C(a, \mathfrak{D}, 1)$. Let w be the unique minimal positive harmonic function in $\mathcal{C}$. Then, for all $x, y \in \mathcal{C}$,*

$$\lim_{t\to\infty} t^{1+\alpha^1} p^{\mathcal{C}}(t, x, y) = \frac{w(x)w(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}, \tag{3.4}$$

where α^1 is the character of $\mathcal{C}$. The limit is in the sense of uniform convergence on compact sets.

PROOF. The proof relies on equation (2.5) and the Dominated Convergence Theorem. Let

$$\begin{aligned} I_1(t) &= \frac{1}{2}\int_0^{t/2}\int_{\mathfrak{S}} \partial_n p^{\mathcal{C}}(s, x, \xi) p^V(t-s, \xi, y)\sigma(d\xi)ds \\ I_2(t) &= \frac{1}{2}\int_{t/2}^{t}\int_{\mathfrak{S}} \partial_n p^{\mathcal{C}}(s, x, \xi) p^V(t-s, \xi, y)\sigma(d\xi)ds. \end{aligned}$$

Then, equation (2.5) and Theorem 2.6 yield,

$$\frac{v(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)} \le \limsup_{t\to\infty} t^{1+\alpha^1} p^{\mathcal{C}}(t, x, y) + \limsup_{t\to\infty} t^{1+\alpha^1}\left(I_1(t) + I_2(t)\right). \tag{3.5}$$

We start by studying $I_1(t)$. For $0 \le s \le t/2$, Theorem 2.6 shows that $t^{1+\alpha^1} p^V(t-s, \xi, y)$ converges to $\frac{v(\xi)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}$. Besides, by using Corollary 3.2 we get the following bound:

$$\begin{aligned} t^{1+\alpha^1} p^V(t-s, \xi, y) &= 2^{1+\alpha^1}(t/2)^{1+\alpha^1} p^V(t-s, \xi, y) \\ &\le 2^{1+\alpha^1} C_1 C_2^{1+|y|}(t/2)^{1+\alpha^1} p^V(t-s, x, x) \\ &\le 2^{1+\alpha^1} C_1 C_2^{1+|y|}\,(t/2)^{1+\alpha^1}\, p^V(t/2, x, x). \end{aligned}$$

The right hand side is uniformly bounded for $t > 1$, so the Dominated Convergence Theorem applies:

$$\lim_{t\to\infty} t^{1+\alpha^1} I_1(t) = \frac{1}{2}\int_0^\infty\int_{\mathfrak{S}} \partial_n p^{\mathcal{C}}(s, x, \xi)\frac{v(\xi)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}\sigma(d\xi)ds = \frac{v(y)\mathbb{E}_x(v(B_{T^{\mathcal{C}}}))}{2^{\alpha^1}\Gamma(1+\alpha^1)}.$$

Next, we study the asymptotics of $I_2(t)$. Since we don't have sharp asymptotics for $\partial_n p^{\mathcal{C}}$ yet, we are not able to use the Dominated Convergence Theorem. Instead, we will resort to Fatou's lemma. For $t/2 \le s \le t$, Lemma 3.3

an Theorem (2.6) imply that

$$\limsup_{t\to\infty} t^{1+\alpha^1}\partial_n p^{\mathcal{C}}(s,x,\xi) \leq \frac{Q_1 v(x)}{2^{\alpha^1}\Gamma(1+\alpha^1)},$$

where Q_1 only depends on x. To show domination, we combine equations (3.1) and (3.3) to get the bound

$$\begin{aligned} t^{1+\alpha^1}\partial_n p^{\mathcal{C}}(s,x,\xi) &\leq t^{1+\alpha^1}\int_{\mathcal{C}} p^V(s-1,x,z)\partial_n p^{\mathcal{C}}(1,z,\xi)dz \\ &\leq t^{1+\alpha^1} p^V(t/2-1,x,x)\int_{\mathcal{C}} C_2^{1+|z|}\partial_n p^{\mathcal{C}}(1,z,\xi)dz. \end{aligned}$$

The right hand side is uniformly bounded in $t>2$ by a constant Q_2 that only depends on x. It follows that

$$\begin{aligned} \limsup_{t\to\infty} t^{1+\alpha^1} I_2(t) &\leq \frac{Q_1 v(x)}{2^{\alpha^1}\Gamma(1+\alpha^1)}\int_0^\infty\int_{\mathfrak{S}} p^V(s,\xi,y)\sigma(d\xi)ds \\ &= \frac{Q_1 v(x)}{2^{\alpha^1}\Gamma(1+\alpha^1)} G^V(\mathfrak{S},y). \end{aligned}$$

Using these two estimates in equation (3.5) we obtain,

$$\frac{v(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)} \leq \limsup_{t\to\infty} t^{1+\alpha^1} p^{\mathcal{C}}(t,x,y) + \frac{v(y)\mathbb{E}_x(v(B_{T^{\mathcal{C}}})) + Q_1 v(x) G^V(\Gamma,y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}.$$

Recall that $w(x) = v(x) - \mathbb{E}_x(v(B_{T^{\mathcal{C}}}))$ for $x\in\mathcal{C}$. Thus,

$$\frac{w(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)} \leq \limsup_{t\to\infty} t^{1+\alpha^1} p^{\mathcal{C}}(t,x,y) + \frac{Q_1 v(x) G^V(\Gamma,y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}. \tag{3.6}$$

We will deduce the theorem from this last estimate. By Corollary 3.2, it is possible to apply Lemma 2.3 to $t^{1+\alpha^1}p^{\mathcal{C}}(t,x,y)$. Therefore, any limit point of this family has the form $\eta w(x)w(y)$ for some $\eta\geq 0$. Different limit points will correspond to different values of η. We will show that this is not the case: by monotonicity of domains, $\eta w(x)w(y) \leq \frac{v(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}$, and since $w(z)$ and $v(z)$ have the same asymptotic behavior for radially convergent $z\to\infty$, we deduce that $\eta \leq \frac{1}{2^{\alpha^1}\Gamma(1+\alpha^1)}$. Let $\eta^* = \sup\eta$, where the supremum is taken over all possible limit points. Equation (3.6) then yields

$$\frac{w(x)v(y)}{2^{\alpha^1}\Gamma(1+\alpha^1)} \leq \eta^* w(x)w(y) + \frac{Q_1 v(x) G^V(\Gamma,y)}{2^{\alpha^1}\Gamma(1+\alpha^1)}.$$

Dividing this equation by $v(y)$ and taking y radially to infinity, the second term on the right hand side vanishes in the limit as $G^V(\Gamma, y)$ is bounded for y away form Γ. We obtain that $\eta^* \geq \frac{1}{2^{\alpha^1}\Gamma(1+\alpha^1)}$, which shows that the only possible limit point is the one given by (3.4). Uniform convergence on compact sets follows from Lemma 2.1. $\square$

4. Asymtotics in multicone domains. We start by fixing $x_0 \in \Omega$, and a sequence (t_k) such that

$$F(x,y) = \lim_{k\to\infty} \frac{p(t_k, x, y)}{p(t_k, x_0, x_0)} = \sum_{i,j=1}^{k} \gamma_{ij} u_i(x) u_j(y),$$

where the converges is uniform in compact sets of $\Omega \times \Omega$, and u_j are the minimal harmonic functions in Ω. This is obtained by a double application of Lemma 2.5. The coefficients $\gamma_{ij} \geq 0$ might depend on the sequence (t_k). Notice that $F(x_0, x_0) = 1$.

By passing to subsequences of (t_k), we can assume that for all $j = 1, \ldots, k$ we have that

$$F_j(x,y) = \lim_{k\to\infty} \frac{p^j(t_k, x, y)}{p(t_k, x_0, x_0)} = \mu_j w_j(x) w_j(y)$$

is well defined. The convergence is uniform in compact sets of $\Omega_j \times \Omega_j$. The coefficient $\mu_j \geq 0$ may also depend on the sequence (t_k).

Our goal is to compute explicitly the coefficients γ_{ij}. In order to do this, we will use equation (2.5) with $O = \Omega$, and $U = \Omega_j$, and estimate the integral involved in (2.5).

It will be convenient to fix points $\xi_j \in \mathfrak{S}_j$, and $z_j \in \Omega_j$. For $x \in \Omega_j$, $y \in \Omega$, $j = 1, \ldots, N$, we define the following object

$$I_j^{x,y}(a, b; t) = \int_a^b \int_{\mathfrak{S}_j} \mathbb{P}_x(B_{T^j} \in d\xi, T^j \in du) p(t-u, \xi, y). \tag{4.1}$$

Most of the technical work of this section will be devoted to find convenient estimates for $I_j^{x,y}$.

We start with a lemma about the function F_j. Recall that $\mathcal{M}$ denotes the set of maximal indices.

Lemma 4.1. *We have that $\mu_j = \mu$ for $j \in \mathcal{M}$, and $\mu_j = 0$ for $j \notin \mathcal{M}$. Also, there is a constant C, independent of the sequence (t_k), such that $\mu \leq C$.*

PROOF. Recall that α_j denotes the character of the branch Ω_j. For $j, l = 1, \dots, N$, and points $x \in \Omega_j$, $y \in \Omega_l$ we have

$$\frac{p^j(t,x,x)}{p(t,x_0,x_0)} = \frac{t^{1+\alpha_j} p^j(t,x,x)}{t^{1+\alpha_l} p^l(t,y,y)} \frac{p^l(t,y,y)}{p(t,x_0,x_0)} \frac{1}{t^{\alpha_j-\alpha_l}}.$$

It follows that

$$\mu_j w_j(x)^2 = \lim_{k\to\infty} \frac{p^j(t_k,x,x)}{p(t_k,x_0,x_0)} = \frac{2^{\alpha_l}\Gamma(1+\alpha_l) w_j(x)^2}{2^{\alpha_j}\Gamma(1+\alpha_j) w_l(y)^2} \mu_l w_l(y)^2 \lim_{t\to\infty} \frac{1}{t^{\alpha_j-\alpha_l}},$$

If $j \notin \mathcal{M}$ and $l \in \mathcal{M}$, we have $\alpha_l < \alpha_j$, and so $\mu_j = 0$. If both $j, l \in \mathcal{M}$, we have $\alpha_j = \alpha_l$, and so $\mu_j = \mu_l = \mu$, only depending on (t_k).

Pick any $j \in \mathcal{M}$. By Harnack's inequality we have

$$\frac{p^j(t_k,z_j,z_j)}{p(t_k+2,x_0,x_0)} \le C_H^2 \frac{p^j(t_k,z_j,z_j)}{p(t_k,z_j,z_j)} \le C_H^2,$$

by monotonicity of domains. Using Lemma 2.2, we see that the left hand side above converges to $\mu w_j(z_j)^2$, thus,

$$\mu \le \frac{C_H^2}{w_j(z_j)^2} \le \frac{C_H^2}{\inf_j w_j(z_j)^2}$$

as desired. $\square$

LEMMA 4.2. *There is a constant $C > 1$ such that, for every $M > 2$, every index $1 \le j \le N$, $m \in \mathcal{M}$, and points $x \in \Omega_j$, $y \in \Omega$ we have for $t_k > 2M+1$*

$$\limsup_k \frac{I_j^{x,y}(0,M;t_k)}{p(t_k,x_0,x_0)} \le C\mathbb{P}_x\left(B_{T^j} \in \mathfrak{S}_j\right) F(\xi_j, y), \tag{4.2}$$

$$\limsup_k \frac{I_j^{x,y}(t_k-M,t_k;t_k)}{p(t_k,x_0,x_0)} \le C\mathbb{1}_{\mathcal{M}}(j) G(\mathfrak{S}_j,y) w_j(x) w_j(z_j), \tag{4.3}$$

$$\limsup_L \limsup_k \frac{I_j^{x,y}(L,t_k-L;t_k)}{p(t_k,x_0,x_0)} \le C\mathbb{1}_{\mathcal{M}}(j) \frac{w_j(x)}{w_m(z)} F(z,y), \tag{4.4}$$

where the last equation holds for any $z \in \Omega_m$. The constant C depends only on the domain Ω and our choices of z_j.

PROOF. Take k large enough such that $t_k > 2M+1$. By the boundary Harnack inequality, there exists a positive C_1 such that for all $u \in [0, t_k - 1]$, all $y \in \Omega$, and all $i = 1, \dots, N$

$$p(t_k-u,\xi,y) \le C_1 p(t_k-u+1,\xi_i,y). \tag{4.5}$$

By Fatou's lemma and Lemma 2.2 we get

$$\overline{\lim_k} \frac{I_j^{x,y}(0,M;t_k)}{p(t_k,x_0,x_0)} \le C_1 \int_0^M \int_{\mathfrak{S}_j} \mathbb{P}_x(B_{T^j} \in d\xi, T^j \in du) F(\xi_j, y),$$

from which (4.2) follows easily.

For $u \in [0,M]$, Lemma 3.3, Theorem (3.4), equation (3.3) and the Dominated Convergence Theorem yield

$$\lim_{k\to\infty} \frac{\partial_n p^j(t_k-u,x,\xi)}{p^j(t_k,x,z_j)} = \frac{\partial_n w_j(\xi)}{w_j(z_j)}, \tag{4.6}$$

where the convergent sequence is bounded by a constant that only depends on M, x and z_j (see Lemma 3.3). It follows by the Dominated Convergence Theorem, and Harnack's inequality, that

$$\begin{aligned}\overline{\lim_k} \frac{I_j^{x,y}(t_k-M,t_k;t_k)}{p(t_k,x_0,x_0)} &\le \overline{\lim_k} \frac{p^j(t_k,x,z_j)}{p(t_k,x_0,x_0)} \int_0^M \int_{\mathfrak{S}_j} \frac{1}{2} \frac{\partial_n w_j(\xi)}{w_j(z_j)} p(u,\xi,y)\sigma(d\xi)du \\ &\le C_2 G(\mathfrak{S}_j,y)\, \overline{\lim_k} \frac{p^j(t_k,x,z_j)}{p(t_k,x_0,x_0)},\end{aligned}$$

where $C_2 = \max\limits_{j=1,\dots,N} \frac{1}{2v_j(z_j)} \sup_{\xi\in\mathfrak{S}_j} \partial_n w_j(\xi)\sigma(\mathfrak{S}_j)$. This inequality and Lemma 4.1 prove (4.3).

Recall that there is $r_0>0$ such that for all $i=1,\dots,N$, we have $B_{2r_0}(\xi_i)\cap \{|x|=1\} \subseteq \mathfrak{S}_i$. For $x \in \Omega_j$, $z\in\Omega_m$, set

$$C_{jm}^L(x,z) = \sup_{t>L} \frac{\int_{\mathfrak{S}_j} \partial_n p^j(t,x,\xi)\sigma(d\xi)}{\int_{\mathfrak{S}_m\cap B_{r_0}(\xi_m)} \partial_n p^m(t,z,\xi')\sigma(d\xi')},$$

which is finite since $m\in\mathcal{M}$. From Lemma 3.3, Theorem 3.4, and the Dominated Convergence Theorem, we obtain

$$\lim_{L\to\infty} C_{jm}^L(x,z) = \mathbb{1}_{\mathcal{M}}(j) \frac{w_j(x)}{w_m(z)} \frac{\int_{\mathfrak{S}_j} \partial_n w_j(\xi)\sigma(d\xi)}{\int_{\mathfrak{S}_m\cap B_{r_0}(\xi_m)} \partial_n w_m(\xi')\sigma(d\xi')} \le C_3 \mathbb{1}_{\mathcal{M}}(j)\frac{w_j(x)}{w_m(z)},$$

for a constant $C_3>0$.

From the standard Harnack inequality we have $p(t-u+1,\xi_m,y) \le C_4 p(t-u+2,\xi',y)$ for all $\xi' \in B_{r_0}(\xi_m)$. The previous discussion yields the following series of inequalities

$$
\begin{aligned}
I_j^{x,y}(L, t_k - L; t_k) &\le C_1 \int_L^{t_k - L} \int_{\mathfrak{S}_j} \frac{1}{2} \partial_n p^j(u, x, \xi) p(t_k - u + 1, \xi_m, y) \sigma(d\xi) du \\
&\le C_1 C_{jm}^L(x, z) \int_L^{t_k - L} \int_{\mathfrak{S}_m \cap B_{r_0}(\xi_m)} \frac{1}{2} \partial_n p^m(u, z, \xi') p(t_k - u + 1, \xi_m, y) \sigma(d\xi') du \\
&\le C_1 C_4 C_{jm}^L(x, z) \int_L^{t_k - L} \int_{\mathfrak{S}_m \cap B_{r_0}(\xi_m)} \frac{1}{2} \partial_n p^m(u, z, \xi') p(t_k - u + 2, \xi', y) \sigma(d\xi') du \\
&\le C_1 C_4 C_{jm}^L(x, z) \int_L^{t_k - L} \int_{\mathfrak{S}_m} \frac{1}{2} \partial_n p^m(u, z, \xi') p(t_k - u + 2, \xi', y) \sigma(d\xi') du \\
&\le C_1 C_4 C_{jm}^L(x, z) p(t_k + 2, z, y).
\end{aligned}
$$

Equation (4.4) now follows from Lemma 2.2. □

LEMMA 4.3. *For the coefficients of the function F defined at the beginning of this section:*

(i) $\gamma_{ij} = 0$ *if* $i \notin \mathcal{M}$ *or* $j \notin \mathcal{M}$.
(ii) *There is a universal constant C depending only on the domain Ω such that $\gamma_{ij} \le C \gamma_{jm}$ for $i, j, m \in \mathcal{M}$, with $i \neq j$.*

PROOF. Let $x \in \Omega_i$ and $y \in \Omega_j$. By (2.5),

$$p(t, x, y) = p^i(t, x, y) \delta_{ij} + I_i^{x,y}(0, t; t).$$

From Lemmas 4.2 and 4.1 we obtain for $m \in \mathcal{M}$, $z \in \Omega_m$,

$$
\begin{aligned}
F(x, y) &\le \delta_{ij} \mu_i w_i(x) w_i(y) + C \left(F(\xi_i, y) + G(\mathfrak{S}_i, y) \mathbb{1}_{\mathcal{M}}(i) w_i(x) w_i(z_i) + \mathbb{1}_{\mathcal{M}}(i) \frac{w_i(x)}{w_m(z)} F(z, y) \right) \\
&\le C \mathbb{1}_{\mathcal{M}}(i) w_i(x) \left(\delta_{ij} w_j(y) + G(\mathfrak{S}_i, y) w_i(z_i) + \frac{F(z, y)}{w_m(z)} \right) + C F(\xi_i, y).
\end{aligned}
$$

The use of this inequality is twofold. First, if $i \notin \mathcal{M}$, by taking x radially to infinity in Ω_i we find that $\gamma_{ij} u_i(x) u_j(y) \le C F(\xi_i, y)$ is only possible if $\gamma_{ij} = 0$. By symmetry of the kernel we conclude (i).

Secondly, consider $i, j, m \in \mathcal{M}$, with $i \neq j$ and $z \in \Omega_m$. Dividing the inequality by $w_i(x) w_j(y)$, and taking $x, y, z \to \infty$ radially in their respective branches, we obtain that $\gamma_{ij} \le C \gamma_{mj}$, as desired. □

REMARK 4.1. *Set $\gamma^* = \max_{i \in \mathcal{M}} \gamma_{ii}$ and fix $m \in \mathcal{M}$ such that $\gamma^* = \gamma_{mm}$. The previous lemma states that $\gamma_{ij} \le C \gamma^*$ for all $i, j \in \mathcal{M}$. Also, notice that*

$$\gamma_{mm} u_m(x) u_m(y) \le F(x, y) \le (1 + C) \gamma_{mm} \sum_{i,j \in \mathcal{M}} u_i(x) u_j(y).$$

Since $F(x_0, x_0) = 1$, we obtain that γ^* is bounded below by a constant that is independent of the sequence (t_k).

It follows that

$$\lim_k \frac{p(t_k, x, y)}{p(t_k, z, z')} = \frac{F(x, y)}{F(z, z')} \leq (1 + C) \sum_{i,j \in \mathcal{M}} \frac{u_i(x) u_j(y)}{u_m(z) u_m(z')}$$

where the constant C comes from Lemma 4.2. In particular, if $x \in \Omega_m$, the inequality

$$\limsup_{t \to \infty} \frac{p(t, x, \xi_m)}{p(t, x, x)} \leq \frac{1 + C}{u_m(x)} \sum_{j \in \mathcal{M}} u_j(\xi_m) \left(1 + \frac{1}{u_m(x)} \sup_{z \in \Omega_m} \sum_{i \neq m} u_i(z) \right) \tag{4.7}$$

holds. If we fix $\widehat{x} \in \mathfrak{D}_m$, and let $r > 0$ be suficiently large, for $x = a_m + r\widehat{x}$ this inequality implies that

$$\limsup_{t \to \infty} \frac{p(t, x, x_0)}{p(t, x, x)} \leq \frac{C_5}{u_m(x)}, \tag{4.8}$$

where $C_5 > 0$ is independent of r.

LEMMA 4.4. *The following inequalities hold*

$$0 < \liminf t^{1+\alpha} p(t, x, y), \quad \limsup t^{1+\alpha} p(t, x, y) < \infty.$$

PROOF. The first inequality is direct from monotonicity of domains and Theorem 3.4 applied to $\Omega_m \subseteq \Omega$.

For the second one, notice that by Harnack's inequaliy, it suffices to prove the theorem for $x = y \in \Omega_m$. We start by setting some constants that will be relevant to our estimates: fix $\widehat{x} \in \mathfrak{D}_m$, and consider $x = a_m + r\widehat{x}$. Then, the Harnack constant

$$C_H = \sup_{s > 1, \xi \in \mathfrak{S}_m} \frac{p(s, \xi, r\widehat{x})}{p(s + 1, \xi_m, r\widehat{x})},$$

is independent of $r > 1$.

Fix $0 < \theta < 1$. In view of (4.8), we can find $x \in \Omega_m$ such that

$$C_H 2^{1+\alpha} \limsup_{t \to \infty} \frac{p(t, \xi_m, x)}{p(t, x, x)} \leq C_H 2^{1+\alpha} \frac{C_5}{u_m(x)} = \frac{\theta}{2}.$$

We fix such an $x = a_m + r\widehat{x}$ from now on. It follows that for large enough t_0, the inequality

$$\frac{p(t,\xi_m,x)}{p(t,x,x)} \leq \frac{\theta}{2^{1+\alpha}C_H}, \qquad \forall\ t > t_0 \tag{4.9}$$

holds.

Next we get estimates using technics somewhat similar to the ones we have used in Lemma 4.2. Let $t > 2t_0$. By Harnack's inequality, and since $t \mapsto p(t,x,x)$ is decreasing

$$\begin{aligned} I_m^{x,x}(0,t/2;t) &\leq C_H \int_0^{t/2} \mathbb{P}_x(B_{T^m} \in \mathfrak{S}_m, T^m \in du) p(t-u+1,\xi_m,x)du \\ &\leq \frac{\theta}{2^{1+\alpha}} \mathbb{P}_x(B_{T^m} \in \mathfrak{S}_m, T^m < t/2) p(t/2,x,x) \\ &\leq \frac{\theta}{2^{1+\alpha}} p(t/2,x,x). \end{aligned}$$

It follows that for all $t > 2t_0$,

$$t^{1+\alpha} I_m^{x,x}(0,t/2;t) \leq \theta \left(\frac{t}{2}\right)^{1+\alpha} p\left(t/2,x,x\right).$$

On the other hand,

$$t^{1+\alpha} I_m^{x,x}(t/2,t;t) \leq \int_0^{t/2} \int_{\mathfrak{S}_m} \frac{t^{1+\alpha}}{2} \partial_n p^m(t-u,x,\xi) p(u,\xi,x) \sigma(d\xi) du.$$

The right hand side converges by (4.6), Theorem 3.4, and the Dominated Convergence Theorem, to

$$C_2 = \frac{w_m(x)}{2^{1+\alpha}\Gamma(1+\alpha)} \int_0^\infty \int_{\mathfrak{S}_m} \partial_n w_m(\xi) p(u,\xi,x)\sigma(d\xi)du < \infty.$$

Putting together these two estimates, we have that, for the continuous function $\varphi(t) = t^{1+\alpha}p(t,x,x)$, $t \geq 2t_0$, it holds that

$$\varphi(t) \leq C_3 + \theta\varphi(t/2), \qquad t > 2t_0,$$

where $C_3 = C_2 + \frac{w_m(x)^2}{2^\alpha\Gamma(1+\alpha)}$. By iteration of the inequality above, it is easy to deduce that, if $t/2^N \in [2t_0, 4t_0]$ then

$$\varphi(t) \leq C_3 \sum_{k=0}^{N-1} \theta^k + \theta^N \varphi(t/2^N) \leq \frac{C_3}{1-\theta} + \sup_{s\in[2t_0,4t_0]} \varphi(s),$$

which finishes the proof. ∎

4.1. *Proof of Theorem 1.1.* The proof is reminiscent of the one we gave for Theorem 3.4. For fixed $x, y \in \Omega$, Lemma 4.4 ensures that $t^{1+\alpha}p(t,x,y)$ is bounded. Harnack's inequality then ensures that the same holds for x, y in any compact set of Ω. This shows that Lemmas 2.1 and 2.3 apply. Let (t_k) be a sequence such that $t_k^{1+\alpha}p(t_k,\cdot,\cdot)$ converges uniformly on compact sets. The limit then has the form $H(x,y) = \sum_{i,j=1}^N \eta_{ij}u_i(x)u_j(y)$, where η_{ij} might depend on the sequence (t_k).

Let $x \in \Omega_m$, and $y \in \Omega$. Set

$$I_1(t) = \frac{1}{2}\int_0^{t/2}\int_{\mathfrak{S}_m} \partial_n p^m(s,x,\xi)p(t-s,\xi,y)\sigma(d\xi)ds,$$
$$I_2(t) = \frac{1}{2}\int_{t/2}^{t}\int_{\mathfrak{S}_m} \partial_n p^m(s,x,\xi)p(t-s,\xi,y)\sigma(d\xi)ds.$$

An application of the Dominated Convergence Theorem, as in the proof of Lemma 4.4, yields that

$$\begin{aligned}\lim_{k\to\infty} t_k^{1+\alpha}I_1(t_k) &= \frac{1}{2}\int_0^{\infty}\int_{\mathfrak{S}_m} \partial_n p^m(s,x,\xi)H(\xi,y)\sigma(d\xi)ds\\ &= \mathbb{E}_x\left(H\left(B_{T^m},y\right)\right).\end{aligned}$$

On the other hand, by using equations (1.3) and (1.4)

$$\begin{aligned}\lim_{k\to\infty} t_k^{1+\alpha}I_2(t_k) &= \mathbb{1}_{\mathcal{M}}(m)\frac{1}{2}\int_0^{\infty}\int_{\mathfrak{S}_m} w_m(x)\partial_n w_m(\xi)p(t-s,\xi,y)\sigma(d\xi)ds\\ &= \mathbb{1}_{\mathcal{M}}(m)\frac{w_m(x)\left(u_m(y)-w_m(y)\right)}{2^\alpha\Gamma(1+\alpha)}.\end{aligned}$$

Using these two estimates, and (2.5), we arrive to

$$H(x,y) = \mathbb{E}_x\left(H\left(B_{T^m},y\right)\right) + \mathbb{1}_{\mathcal{M}}(m)\frac{w_m(x)u_m(y)}{2^\alpha\Gamma(1+\alpha)}, \qquad x \in \Omega_m, y \in \Omega.$$

Recall that $\mathbb{E}_x\left(H\left(B_{T^m},y\right)\right)$ is bounded as function of x. By taking the limit of $H(x,y)/u_m(x)$, with x going radially to infinity, we find that

$$\sum_{j=1}^N \eta_{mj}u_j(y) = \mathbb{1}_{\mathcal{M}}(m)\frac{u_m(y)}{2^\alpha\Gamma(1+\alpha)},$$

By uniqueness of the decomposition (2.6), we find that the only nonzero coefficients are $\gamma_{mm} = \frac{1}{2^\alpha\Gamma(1+\alpha)}$ for $m \in \mathcal{M}$. This shows (1.6). Uniform convergence on compact sets is direct from Lemma 2.1. ☐

The following corollary is a direct consecuence of the previous theorem.

Corollary 4.5. *Let Ω be a mutlticone domain, with maximal index set $\mathcal{M}$. Then,*

$$\lim_{t\to\infty} \frac{p(t,x,y)}{p(t,w,z)} = \frac{\sum_{j\in\mathcal{M}} u_j(x)u_j(y)}{\sum_{j\in\mathcal{M}} u_j(w)u_j(z)}. \tag{4.10}$$

The convergence is uniform in compact sets.

5. Asymptotics for the exit time. The following result is taken form [3].

Theorem 5.1. *Let $V \subseteq \mathbb{R}^n$ be a cone with vertex 0 and opening $\mathfrak{D}$. Assume that $\mathfrak{D}$ is regular por the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$, and let α be the character of $\mathfrak{D}$. Set $\kappa = 1+\alpha-n/2$, and let T^V be the Brownian exit time from V. Then, for each $x \in V$,*

$$\lim_{t\to\infty} t^{\kappa/2}\mathbb{P}_x(T^V > t) = \gamma_V v(x). \tag{5.1}$$

Here $v(x) = |x|^\kappa m^1(x/|x|)$ is the harmonic function defined in (1.1), where m^1 is the only non-negative eigenfunction of the Laplace-Beltrami operator on $\mathfrak{D}$ with Dirichlet boundary conditions. Also,

$$\gamma_V = \frac{\Gamma\left(\frac{\kappa+n}{2}\right)}{2^{\kappa/2}\Gamma\left(\kappa+\frac{n}{2}\right)} \int_{\mathfrak{D}} m^1(\theta)\sigma(d\theta).$$

Remark 5.1. *From this theorem, the scaling property of the heat kernel in V, and Harnack's inequality up to the boundary, we get the following bound:*

$$\begin{aligned} t^{\kappa/2}\mathbb{P}_x(T^V > t) &= t^{\kappa/2}\mathbb{P}_{x/|x|}\left(T^V > \frac{t}{|x|^2}\right) \leq C_H t^{\kappa/2}\mathbb{P}_\xi\left(T^V > \frac{t}{|x|^2}\right) \\ &\leq C_H |x|^\kappa \left(\frac{t}{|x|^2}\right)^{\kappa/2} \mathbb{P}_\xi\left(T^V > \frac{t}{|x|^2}\right). \end{aligned}$$

where $\xi \in \mathfrak{D}$ is fixed. Using (5.1), we can pick t_ξ such that whenever $t/|x|^2 > t_\xi$, the right hand side of the last display is bounded by $C|x|^\kappa$, where C depends on our choice of ξ. For $t/|x|^2 \leq t_\xi$, we have that $t^{\kappa/2} \leq t_\xi^{\kappa/2}|x|^\kappa$. We deduce that there is a universal constant $C > 0$ such that

$$t^{\kappa/2}\mathbb{P}_x(T^V > t) \leq C|x|^\kappa, \qquad x \in V,\ t > 0. \tag{5.2}$$

By monotoncity of domains, the same inequality holds for $T^{\mathcal{C}}$, and $x \in \mathcal{C}$, where $\mathcal{C}$ is any truncated cone with opening $\mathfrak{D}$.

The following lemma will be the key tool when extending the previous result to multicones.

LEMMA 5.2. *Let T be the exit time from a multicone set Ω, let T^j be the exit time from Ω_j, and pick $x \in \Omega_i$ for some $i = 1, \dots, N$. We have*

$$\mathbb{P}_x(T > t) = \mathbb{P}_x(T^i > t) + \mathbb{P}_x(B_t \in \Omega_0, T > t) + \tag{5.3}$$
$$+ \frac{1}{2} \sum_{j=1}^{N} \int_0^t \int_{\mathfrak{S}_j} \partial_n \mathbb{P}_z(T^j > t - s) p(s, x, z) \sigma(dz) ds.$$

PROOF. For $j = 1, \dots, k$, and $0 \le s \le t$ define the functions

$$f_j(s) = \int_{\Omega_j} \mathbb{P}_z(T^j > t - s) p(s, x, z) dz.$$

For $s < t$, since $u(s, z) = \mathbb{P}_z(T^j > s)$, and $v(s, z) = p(s, x, z)$ are solutions of the heat equation with dirichlet boundary condition in Ω_j and Ω respectively, we have by Green's formula

$$\begin{aligned} \frac{df_j(s)}{ds} &= \frac{1}{2} \int_{\Omega_j} -p(s, x, z) \Delta_z \mathbb{P}_z(T^j > t - s) + \mathbb{P}_z(T^j > t - s) \Delta_z p(s, x, z) dz \\ &= \frac{1}{2} \int_{\mathfrak{S}_j} p(s, x, z) \partial_n \mathbb{P}_z(T^j > t - s) - \mathbb{P}_z(T^j > t - s) \partial_n p(s, x, z) \sigma(dz) \\ &= \frac{1}{2} \int_{\mathfrak{S}_j} p(s, x, z) \partial_n \mathbb{P}_z(T^j > t - s) \sigma(dz), \end{aligned}$$

where, as usual, ∂_n represents the (inward) normal derivative. Then, for every $\varepsilon > 0$,

$$f_j(t - \varepsilon) - f_j(0) = \frac{1}{2} \int_0^{t-\varepsilon} \int_{\mathfrak{S}_j} p(s, x, z) \partial_n \mathbb{P}_z(T^j > t - s) \sigma(dz). \tag{5.4}$$

In order to extend this equation to $\varepsilon = 0$, we need an estimate for $\partial_n \mathbb{P}_z(T^j > u)$ for u near zero. The process B leaves Ω_j before the norm of B hits level 1. Since $\rho_t = |B_t|$ is a Bessel process, if we let τ be the hitting time of 1 for an $n-$dimensional Bessel process, then, for $t < \tau$,

$$\rho_t = |z| + \beta_t + \frac{n-1}{2} \int_0^t \rho_s^{-1} ds \le |z| + \beta_t + \frac{n-1}{2} t,$$

for some one dimensional Brownian motion β_t. Let τ^μ be the hitting time of zero for the Brownian motion with drift $\beta_t + \mu t$, with $\mu = \frac{n-1}{2}$. It follows that

$$\mathbb{P}_z(T^j > u) \le \mathbb{P}_{|z|}(\tau > u) \le \mathbb{P}_r(\tau^\mu > u), \qquad r = |z| - 1.$$

As $\mathbb{P}_z(T^j > u)$ vanishes on $\mathfrak{S}_j$, we get $\partial_n \mathbb{P}_z(T^j > u) \le \partial_r \mathbb{P}_r(\tau^\mu > u)|_{r=0}$. The distribution of τ^μ is well known (see equation (5.12), pp 197 [6]):

$$\mathbb{P}_r(\tau^\mu \in dt) = \frac{r}{\sqrt{2\pi t^3}} \exp\left[-\frac{(r+\mu t)^2}{2t}\right] dt, \qquad t > 0.$$

A direct computation shows that

$$\begin{aligned}
\mathbb{P}_r(\tau^\mu > u) &= \int_u^\infty \frac{r}{\sqrt{2\pi t^3}} \exp\left[-\frac{(r+\mu t)^2}{2t}\right] dt, \\
\partial_r \mathbb{P}_r(\tau^\mu > u)|_{r=0} &= \int_u^\infty \frac{1}{\sqrt{2\pi t^3}} \exp\left[-\frac{\mu^2 t}{2}\right] dt \\
&\le \int_u^1 \frac{t^{-3/2}}{\sqrt{2\pi}} dt + \int_1^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\mu^2 t}{2}\right] dt \\
&= \sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{u}} - 1\right) + \sqrt{\frac{2}{\pi}} \mu^{-2} e^{-\mu^2/2}, \qquad u < 1,
\end{aligned}$$

which is integrable (in u) near zero. Therefore, we can apply the Dominated Convergence Theorem in (5.4) and use the continuity of f_j to deduce that this equation also holds for $\varepsilon = 0$. Adding all the equations for $j = 1, \ldots, N$, using that $f_j(0) = \mathbb{1}_{\Omega_j}(x)\mathbb{P}_x(T^j > t)$ and $f_j(t) = \mathbb{P}_x(B_t \in \Omega_j, T > t)$, and adding the contribution from Ω_0, we obtain (5.3). $\square$

5.1. *Proof of Theorem 1.2 for truncated cones.* Theorem 5.1 is also valid if we change the cone $V = C(a, \mathfrak{D}, 0)$ for its truncated version $\mathcal{C} = C(a, \mathfrak{D}, 1)$, but the limit turns out to be $\gamma_V w(x)$, where $w(\cdot)$ is the unique positive harmonic function in $\mathcal{C}$ that vanishes on $\partial\mathcal{C}$, normalized at infinity, such that $\lim_{r\to\infty} r^{-\kappa} w(a + r\theta) = 1$ for any $\theta \in \mathfrak{D}$. Recall that $\mathfrak{S} = a + \mathfrak{D}$ is the base of $\mathcal{C}$.

Indeed, notice that $(t, x) \mapsto \mathbb{P}_x(T^{\mathcal{C}} > t)$ solves the heat equation in $\mathcal{C}$. By Harnack's inequality, the family of functions $h(t, x) = t^{\kappa/2}\mathbb{P}_x(T^{\mathcal{C}} > t)$, indexed by $t > 0$, is bounded on compact sets of $\mathcal{C}$. Since $\mathbb{P}_x(T^{\mathcal{C}} > t) \le \mathbb{P}_x(T^V > t)$, Lemma 2.3 applies and we conclude that any limit point has the form $\mu w(x)$. Of course, the constant μ may depend on the sequence (t_k) that makes $h(t_k, \cdot)$ converge. Nevertheless, we have $\mu \le \gamma_V$ by monotonicity of domains, where γ_V is the same constant as in Theorem 5.1.

On the other hand, from Lemma 5.2, we have that for all $x \in \mathcal{C}$

$$\begin{aligned}
\mathbb{P}_x(T^V > t) = \mathbb{P}_x(T^{\mathcal{C}} > t) + \mathbb{P}_x(|B_t| < 1, T^V > t) + \\
+ \frac{1}{2}\int_0^t \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > t - s) p^V(s, x, z)\sigma(dz) ds.
\end{aligned}$$

Also, by Harnack's inequality $t^{\kappa/2}\mathbb{P}_x(|B_t| < 1, T^V > t) \le C_1 t^{\kappa/2} p(t+1, x, x_0)$, which converges to zero, as $t \to \infty$ by Theorem 1.1. Thus,

$$\gamma_V v(x) \le \mu w(x) + \varlimsup_{t\to\infty} \frac{t^{\kappa/2}}{2} \int_0^t \int_{\mathfrak{S}_j} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > t-s) p^V(s,x,z)\sigma(dz)ds. \tag{5.5}$$

We will next show how to control the integral on the right hand side of (5.5).

By applying Fubini's theorem to the Chapman-Kolmogorov equation for the heat kernel, we get for $t, s > 0$,

$$\mathbb{P}_x(T^{\mathcal{C}} > t+s) = \int_{\mathcal{C}} p^{\mathcal{C}}(s,x,z)\mathbb{P}_z(T^{\mathcal{C}} > t)dz.$$

Using the heat kernel of the exterior of a ball we get the upper bound $\partial_x p^{\mathcal{C}}(s,x,z) \le Ae^{-B|z|^2}$ for $s > 1$ and all $x \in \mathcal{C}$. We can apply the Dominated Convergence Theorem to get for $x \in \mathfrak{S}$

$$\partial_n \mathbb{P}_x(T^{\mathcal{C}} > t+s) = \int_{\mathcal{C}} \partial_n p^{\mathcal{C}}(s,x,z)\mathbb{P}_z(T^{\mathcal{C}} > t)dz. \tag{5.6}$$

Moreover, using (5.2), it is easy to obtain the following limit by using again the Dominated Convergence Theorem

$$\lim_k t_k^{\kappa/2}\partial_n \mathbb{P}_x(T^{\mathcal{C}} > t_k) = \mu \int_{\mathcal{C}} \partial_n p^{\mathcal{C}}(s,x,z)w(z)dz = \mu\partial_n w(x).$$

The last equality holds because w is harmonic.

As usual, we split the integral from (5.5) into:

$$\begin{aligned} I_1(t) &= \int_0^{t/2} \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > t-s)p^V(s,x,z)\sigma(dz)ds \\ &\le C_H \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > t/2)\sigma(dz) \int_0^{t/2} p^V(s+1,x,\xi)ds. \end{aligned}$$

This shows that $\varlimsup_{t\to\infty} t^{\alpha/2} I_1(t) \le C_1 G^V(x,\xi)$, where $C_1 > 0$ is universal.

Also, by the boundary Harnack inequality

$$\begin{aligned} I_2(t) &= \int_{t/2}^t \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > t-s)p^V(s,x,z)\sigma(dz)ds \\ &\le C_H \int_{t/2}^t \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > t-s)\sigma(dz)ds \; p^V(t/2,x,x) \\ &\le C_x t^{-1-\alpha} \int_0^{t/2} \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > s)\sigma(dz)ds \end{aligned}$$

where C_x only depends on x. Using bounds for the exit time for a Bessel process from $[1,\infty)$ as in Lemma 5.2, we get that $\int_0^1 \partial_n \mathbb{P}_z(T^{\mathcal{C}} > s)ds \le Q$, independently of $z \in \mathfrak{S}$. Then

$$t^{\kappa/2} I_2(t) \le C_x t^{-1-\alpha+\kappa/2} \left(Q\,|\mathfrak{S}| + \left(\frac{t}{2} - 1\right) \int_{\mathfrak{S}} \partial_n \mathbb{P}_z(T^{\mathcal{C}} > 1)\sigma(dz) \right).$$

It follows that $t^{\kappa/2} I_2(t) \to 0$ as $t \to \infty$. Equation (5.5) now reads

$$\gamma_V v(x) \le \mu w(x) + C_1 G^V(\xi, x), \qquad x \in \mathcal{C}.$$

Since $G^V(\xi,x)$ remains bounded as $x \to \infty$ radially in $\mathcal{C}$, we deduce that $\gamma_V = \mu$, which proves the asymptotic for the survival probability. □

5.2. *Proof of Theorem 1.2.* In formula (5.3), the first term is controlled by our result from the previous section. The second term goes to zero by using Harnack's inequality up to the boundary, that is, for some $x_0 \in \Omega_0$,

$$\mathbb{P}_x(B_t \in \Omega_0, T > t) = \int_{\Omega_0} p(t,x,z)dz \le C_H\,|\Omega_0|\,p(t+1,x,x_0),$$

where $|\Omega_0|$ stands for the Lebesgue measure of the core Ω_0. It follows that $t^{\kappa/2}\mathbb{P}_x(B_t \in \Omega_0, T > t)$ converges to zero as $t \to \infty$ for each $x \in \Omega$.

Next, we deal with the summation terms. In order to do this, we will find limits for the following two objects:

$$t^{\kappa/2} I_1(t) = t^{\kappa/2} \int_0^{t/2} \int_{\mathfrak{S}_j} \partial_n \mathbb{P}_z(T^j > t-s) p(s,x,z)\sigma(dz)ds,$$
$$t^{\kappa/2} I_2(t) = t^{\kappa/2} \int_0^{t/2} \int_{\mathfrak{S}_j} \partial_n \mathbb{P}_z(T^j > s) p(t-s,x,z)\sigma(dz)ds.$$

An analogous proof as the one in the last part of the previous section, shows that $t^{\kappa/2} I_2(t)$ converges to zero.

As for $t^{\kappa/2} I_1(t)$, if $0 \le s \le t/2$, our computations in the previous section show that $t^{\kappa/2}\partial_n \mathbb{P}_z(T^j > t-s)$ converges to $\gamma_j \partial_n w_j(z)$ for $z \in \mathfrak{S}_j$ and $j \in \mathcal{M}$, otherwise, it converges to zero. Here,

$$\gamma_j = \frac{\Gamma\left(\frac{\kappa+n}{2}\right)}{2^{\kappa/2}\Gamma\left(\kappa + \frac{n}{2}\right)} \int_{\mathfrak{S}_j} m_j^1(\theta)\sigma(d\theta).$$

To show domination, we use monotonicity of $t \mapsto \partial_n \mathbb{P}_x(T^j > t)$, equation (5.6), the bound $\partial_n p^{\mathcal{C}}(1,x,z) \le Ae^{-B|z|^2}$, and equation (5.2). We find that

$$t^{\kappa/2}\partial_n \mathbb{P}_x(T^j > t-s) \le C_3 \int_{\mathcal{C}} Ae^{-B|z|^2}\,|z|^{\kappa}\,dz < \infty.$$

Thus, by the Dominated Convergence Theorem, we deduce that

$$\begin{aligned}\lim_{t\to\infty} t^{\kappa/2} I_1(t) &= \gamma_j \int_0^\infty \int_{\mathfrak{S}_j} \partial_n w_j(z) p(s,x,z)\sigma(dz)ds \\ &= \gamma_j \int_{\mathfrak{S}_j} \partial_n w_j(z) G(x,z)\sigma(dz) = 2\gamma_j \left(u_j(x) - w_j(x)\right),\end{aligned}$$

by Fubini's theorem, and equation (1.3).

Putting all together, for $x \in \Omega$,

$$\lim_{t\to\infty} t^{\kappa/2}\mathbb{P}_x(T > t) = \sum_{k\in\mathcal{M}} \gamma_k u_k(x),$$

which is (1.7). □

6. Renormalized Yaglom limit for multicones. In what follows, we set $\beta = 1 + \alpha + n/2$. Notice that $\beta/2 + \kappa/2 = 1 + \alpha$, which will be conveniently used later.

From Theorem 2.6, it is straightforward to get that for $x, y \in V = C(0, \mathfrak{D}, 0)$

$$\lim_{t\to\infty} t^{\beta/2} p^V(t,x,\sqrt{t}y) = \frac{v(x)v(y)}{2^\alpha \Gamma(1+\alpha)} e^{-|y|^2/2}. \tag{6.1}$$

The limit above holds uniformly in compact sets of $\overline{V}$.

In order to extend this result to multicones, we start with the case of a truncated cone $\mathcal{C} = C(0, \mathfrak{D}, 1)$.

LEMMA 6.1. *Let $x, y \in \mathcal{C}$. Then,*

$$\lim_{t\to\infty} t^{\beta/2} p^{\mathcal{C}}(t,x,\sqrt{t}y) = \frac{w(x)v(y)}{2^\alpha \Gamma(1+\alpha)} e^{-|y|^2/2}. \tag{6.2}$$

PROOF. As in the proof of Lemma 3.3, we have that $t^{1+\alpha}\partial_n p^{\mathcal{C}}(t,x,\xi) \le Q_x$ for all $t > t_0$, $\xi \in \mathfrak{S}$. Thus, using the boundary Harnack inequality, there exist $C_x > 0$ only dependent on x, such that

$$\frac{t^{\beta/2}}{2} \int_0^{t/2} \int_{\mathfrak{S}} \partial_n p^{\mathcal{C}}(t-s,x,\xi) p^V(s,\xi,\sqrt{t}y) d\xi ds \le C_x G^V(x,\sqrt{t}y) t^{\beta/2-(1+\alpha)}. \tag{6.3}$$

For large t, the quantity $G^V(x, \sqrt{t}y)$ is bounded, and as $\beta/2 - (1+\alpha) = -\kappa/2 < 0$, we deduce that

$$\lim_{t\to\infty} \frac{t^{\beta/2}}{2} \int_0^{t/2} \int_{\mathfrak{S}} \partial_n p^{\mathcal{C}}(t-s, x, \xi) p^V(s, x, \sqrt{t}y) d\xi ds = 0. \tag{6.4}$$

On the other hand, from Theorem 2.6, it is direct to find the bound

$$t^{\beta/2} p^V(t, x, \sqrt{t}y) \le C \sum_{i=1}^{\infty} \frac{(|x|\,|y|)^{\alpha^i - (\frac{n}{2}-1)}}{2^{\alpha^i}\Gamma(1+\alpha^i)} < \infty, \tag{6.5}$$

for $t > 2$ and some universal constant $C > 0$. It follows by (6.1) and the Dominated Convergence Theorem that

$$\begin{aligned} \lim_{t\to\infty} \frac{t^{\beta/2}}{2} \int_0^{t/2} \int_{\mathfrak{S}} \partial_n p^{\mathcal{C}}(s, x, \xi) p^V(t-s, \xi, \sqrt{t}y) d\xi ds = \\ = \frac{v(y)e^{-|y|^2/2}}{2^\alpha \Gamma(1+\alpha)} \int_0^\infty \int_{\mathfrak{S}} \frac{1}{2} \partial_n p^{\mathcal{C}}(s, x, \xi) v(\xi) d\xi ds \\ = e^{-|y|^2/2} \frac{v(y)\mathbb{E}_x(v(B_{T^{\mathcal{C}}}))}{2^\alpha \Gamma(1+\alpha)} \end{aligned} \tag{6.6}$$

Plugging the last two equations into (2.5), with $O = V$ and $U = \mathcal{C}$, we obtain

$$\lim_{t\to\infty} t^{\beta/2} p^{\mathcal{C}}(t, x, \sqrt{t}y) = \lim_{t\to\infty} t^{\beta/2} p^V(t, x, \sqrt{t}y) - e^{-|y|^2/2} \frac{v(y)\mathbb{E}_x(v(B_{T^{\mathcal{C}}}))}{2^\alpha \Gamma(1+\alpha)},$$

from where (6.2) is direct to deduce by using (1.2). □

Lemma 6.2. *We have for each $y \in V$*

$$\sup_{t>2} \sup_{\xi \in \mathfrak{S}} t^{\beta/2} \partial_n p^{\mathcal{C}}(t, \sqrt{t}y, \xi) < \infty \tag{6.7}$$

Also, for $y \in \mathcal{C}$ and $\xi \in \mathfrak{S}$,

$$\lim_{t\to\infty} t^{\beta/2} \partial_n p^{\mathcal{C}}(t, \sqrt{t}y, \xi) = \frac{\partial_n w(\xi) v(y)}{2^\alpha \Gamma(1+\alpha)} e^{-|y|^2/2}. \tag{6.8}$$

The limit holds in the sense of uniform convergence in compact sets.

PROOF. Recall that, from the bound for the heat kernel of the exterior of a ball, and monotonicity of domains

$$t^{\beta/2}\partial_n p^{\mathfrak{C}}(t+1,\sqrt{t}y,\xi) = t^{\beta/2}\int_{\mathfrak{C}} \partial_n p^{\mathfrak{C}}(1,z,\xi) p^{\mathfrak{C}}(t,\sqrt{t}y,z)dz \tag{6.9}$$

$$\leq t^{\beta/2}\int_{\mathfrak{C}} Ae^{-B|z|^2} p^V(t,\sqrt{t}y,z)dz \tag{6.10}$$

$$\leq AC\int_{\mathfrak{C}} e^{-B|z|^2}\sum_{i=1}^{\infty}\frac{(|z|\,|y|)^{\alpha^i-\left(\frac{n}{2}-1\right)}}{2^{\alpha^i}\Gamma(1+\alpha^i)} \tag{6.11}$$

where the last inequality follows from (6.5). From here, using bounds for the moment of gaussian random variables, we arrive at the bound:

$$t^{\beta/2}\partial_n p^{\mathfrak{C}}(t+1,\sqrt{t}y,\xi) \leq C_6\sum_{i=1}^{\infty}\frac{C_7^{\alpha^i+n/2}\,|y|^{\alpha^i-\left(\frac{n}{2}-1\right)}}{2^{\alpha^i/2}\Gamma(1+\alpha^i)}\Gamma\left(\frac{1+\alpha^i}{2}+\frac{n}{4}\right), \tag{6.12}$$

which is finite.

The same steps as above show that it is possible to apply the Dominated Convergence Theorem in (6.9). Equation (6.8) then follows from Lemma 6.1. □

6.1. *Proof of Theorem 1.3.*

PROOF. As before, our starting point is equation (2.5). We will study the rate of decay of the integral involved in such equation by splitting in two terms, as before. First, let us study

$$t^{\beta/2}I_1(t) = t^{\beta/2}\int_0^{t/2}\int_{\mathfrak{S}_j}\frac{1}{2}\partial_n p^j(t-s,a_j+\sqrt{t}y,\xi)p(s,x,\xi)d\xi ds. \tag{6.13}$$

Using Lemma (6.2), we see that we can apply the Dominated Convergence Theorem to this integral to obtain

$$\begin{aligned}\lim_{t\to\infty} t^{\beta/2}I_1(t) &= \mathbb{1}_{\mathcal{M}}(j)\int_0^{\infty}\int_{\mathfrak{S}_j}\frac{1}{2}\frac{\partial_n w_j(\xi)e^{-|y|^2/2}v_j(y)}{2^{\alpha}\Gamma(1+\alpha)}p(s,x,\xi)d\xi ds\\ &= \mathbb{1}_{\mathcal{M}}(j)\frac{e^{-|y|^2/2}v_j(y)}{2^{\alpha}\Gamma(1+\alpha)}\int_{\mathfrak{S}_j}\frac{1}{2}\partial_n w_j(\xi)G(x,\xi)d\xi ds\\ &= \mathbb{1}_{\mathcal{M}}(j)\frac{e^{-|y|^2/2}v_j(y)}{2^{\alpha}\Gamma(1+\alpha)}\left(u_j(x)-w_j(x)\right).\end{aligned} \tag{6.14}$$

Second, we look at

(6.15)
$$\begin{aligned} t^{\beta/2} I_2(t) &= t^{\beta/2} \int_0^{t/2} \int_{\mathfrak{S}_j} \frac{1}{2} \partial_n p^j(s, a_j + \sqrt{t}y, \xi) p(t-s, x, \xi) d\xi ds \\ &= t^{-\kappa/2} \int_0^{t/2} \int_{\mathfrak{S}_j} \frac{1}{2} \partial_n p^j(s, a_j + \sqrt{t}y, \xi) t^{1+\alpha} p(t-s, x, \xi) d\xi ds \end{aligned}$$

From Theorem 1.1, we have that $t^{1+\alpha} p(t-s, x, \xi) \leq C_x$ for all $s \in [0, t/2]$ as long as $t > 3$. The constant C_x depends only on $|x|$. Then

$$\begin{aligned} t^{\beta} I_2(t) &\leq C_x t^{-\kappa/2} \int_0^{t/2} \int_{\mathfrak{S}_j} \frac{1}{2} \partial_n p^j(s, a_j + \sqrt{t}y, \xi) d\xi ds \\ &= C_x t^{-\kappa/2} \mathbb{P}_{\sqrt{t}y}(B_{T^j} \in \mathfrak{S}_j, T^j < t/2) \leq C_x t^{-\kappa/2}, \end{aligned}$$

which converges to zero.

Putting together equation (2.5), Lemma 6.1, and the last estimates, we obtain

$$\lim_{t \to \infty} t^{\beta/2} p(t, x, a_j + \sqrt{t}y) = \mathbb{1}_{\mathcal{M}}(j) \frac{u_j(x) v_j(y)}{2^\alpha \Gamma(1+\alpha)} e^{-|y|^2/2}, \tag{6.16}$$

as desired. ☐

6.2. *Distributional convergence of the renormalized process.* Theorem 1.3 suggests that, when conditioned on survival, most of the trajectories of Brownian motion at time t stay within order $\sqrt{t}$ from the origin. Thus, it is natural to study the convergence of the rescaled process $B_t/\sqrt{t}$ conditioned on survival.

Let $A \subseteq V_j$ be a precompact, Borel set. Notice that $\beta - \kappa = n$. Then, by a simple change of variable

$$\begin{aligned} \mathbb{P}_x\Big((B_t - a_j)/\sqrt{t} \in A | T > t\Big) &= \int_{\sqrt{t}A} \frac{p(t, x, a_j + z)}{\mathbb{P}_x(T > t)} dz \\ &= \int_A \frac{p(t, x, a_j + \sqrt{t}y)}{\mathbb{P}_x(T > t)} t^{n/2} dy \\ &= \int_A \frac{t^{\beta/2} p(t, x, a_j + \sqrt{t}y)}{t^{\kappa/2} \mathbb{P}_x(T > t)} dy, \end{aligned}$$

By Theorems 1.2 and 1.3, the integrand on the right hand side converges to the function

$$p_x(j,y) = \frac{v_j(y)e^{-|y|^2/2}}{\gamma_j 2^\alpha \Gamma(1+\alpha)} \cdot \frac{\gamma_j u_j(x)}{\sum_{k\in\mathcal{M}} \gamma_k u_k(x)} \mathbb{1}_{V_j}(y). \tag{6.17}$$

Equation (6.17) defines a probability distribution function on $\mathcal{M} \times \cup_{j\in\mathcal{M}} V_j$ for a family of random variables $X^x = (X_1^x, X_2^x)$, with $x \in \Omega$, which is simple to interpret. Fix $x \in \Omega$ and let X_1^x be a discrete random variable with distribution given by

$$\mathbb{P}(X_1^x = j) = \frac{\gamma_j u_j(x)}{\sum_{k\in\mathcal{M}} \gamma_k u_k(x)}, \qquad j \in \mathcal{M}. \tag{6.18}$$

This is a sample of one of the maximal branches of the multicone. As $t \to \infty$ the multicone Ω_j is rescaled into the cone with vertex V_j. Correspondingly, we define X_2^x as a continuous random variable on $\cup_{j\in\mathcal{M}} V_j$ satisfying

$$\mathbb{P}(X_2^x \in dy | X_1^x = j) = \frac{v_j(y)e^{-|y|^2/2}}{\gamma_j 2^\alpha \Gamma(1+\alpha)} \mathbb{1}_{V_j}(y). \tag{6.19}$$

Our computation at the beginning of the section, and the uniform convergence on compact sets shows that, under $\mathbb{P}_x$, the renormalized process $B_t/\sqrt{t}$ conditioned on survival converges weakly to X^x.

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